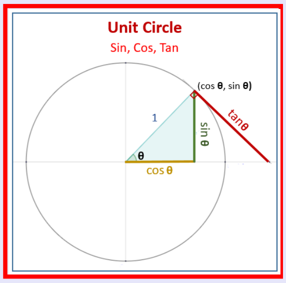
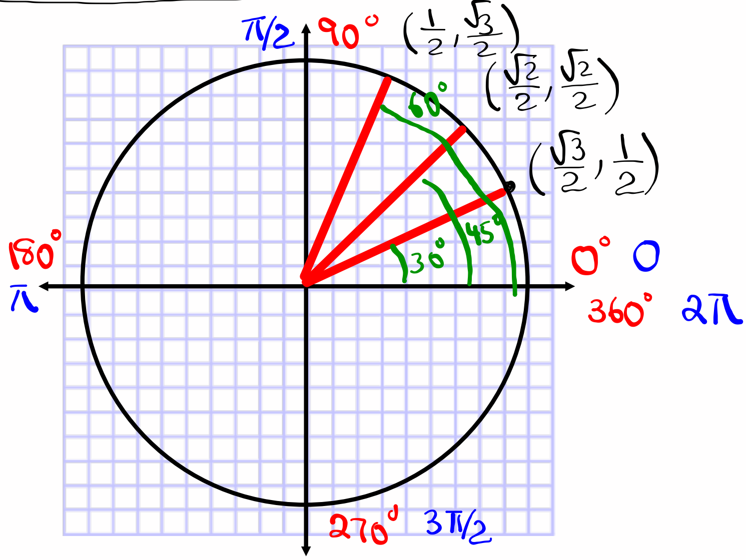
Math 241

Winter 2023

Lecture 8



x	0° 0	30° $\pi/6$	45° $\pi/4$	60° $\pi/3$	90° $\pi/2$	180° π	270° $3\pi/2$	360° 2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0



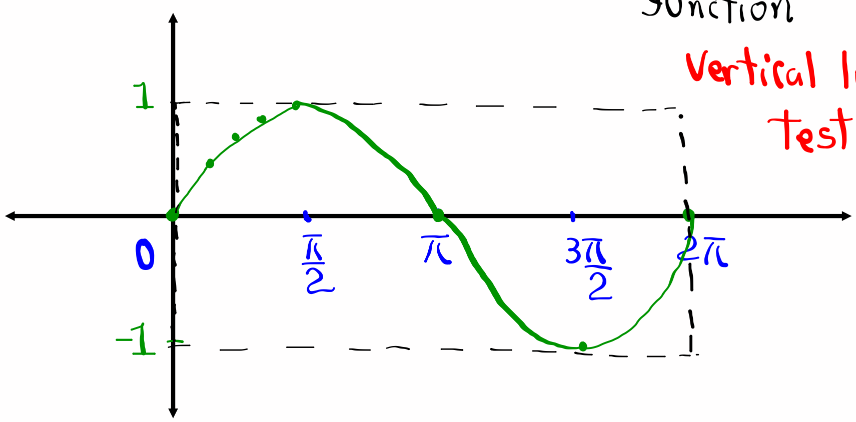
Graphing $y = \sin x$

One period $0 \leq x \leq 2\pi$

Range $-1 \leq \sin x \leq 1$

$y = \sin x$
function

Vertical line test

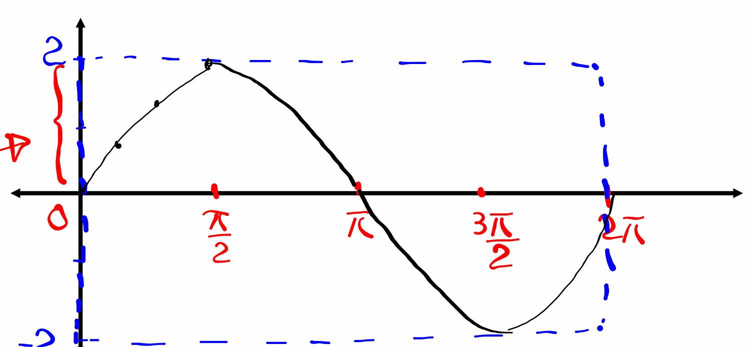


Graph $y = 2\sin x$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$2\sin x$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	0	-2	0

$-1 \leq \sin x \leq 1$

$-2 \leq 2\sin x \leq 2$



Amplitude = 2

Graph $y = -4\sin x$

$0 \le x \le 2\pi$

one period

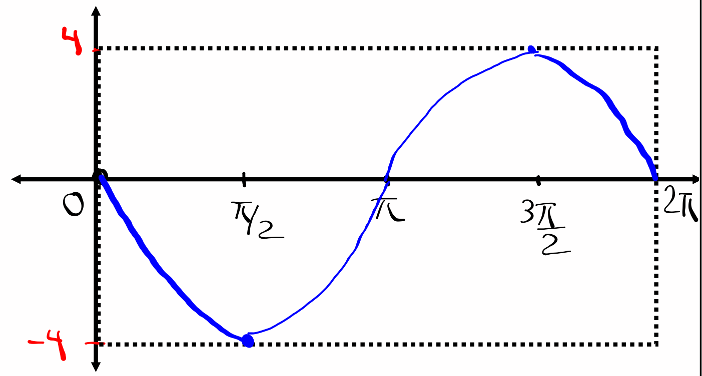
$-1 \le \sin x \le 1$

$4 \ge -4\sin x \ge -4$

$-4 \le 4\sin x \le 4$

Amplitude = 4

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$-4\sin x$	0	-2	$-2\sqrt{2}$	$-2\sqrt{3}$	-4	0	4	0



Graph $y = 1\sin\left(x - \frac{\pi}{2}\right)$

$0 \le x \le 2\pi$

$0 \le x - \frac{\pi}{2} \le 2\pi$

$\frac{\pi}{2} \le x \le 2\pi + \frac{\pi}{2}$

$\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

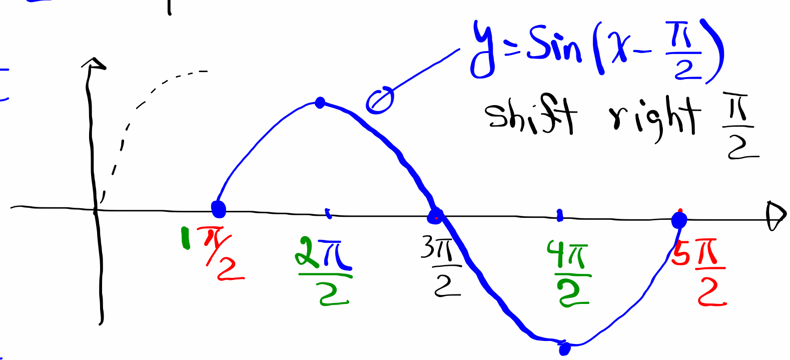
$\frac{\pi}{2} + \frac{5\pi}{2} = \frac{6\pi}{2} = 3\pi$

$\frac{3\pi}{2}$

$\frac{\pi}{2} + \frac{3\pi}{2} = \frac{4\pi}{2} = 2\pi$

$\frac{2\pi}{2} = \pi$

x	0	$\pi/2$	π	$3\pi/2$	2π
$x - \frac{\pi}{2}$	$-\pi/2$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin(x - \pi/2)$			0	1	0



Graph $y = \underset{A}{3}\sin(x+\pi)$

$0 \le x \le 2\pi$

$0 \le x+\pi \le 2\pi$

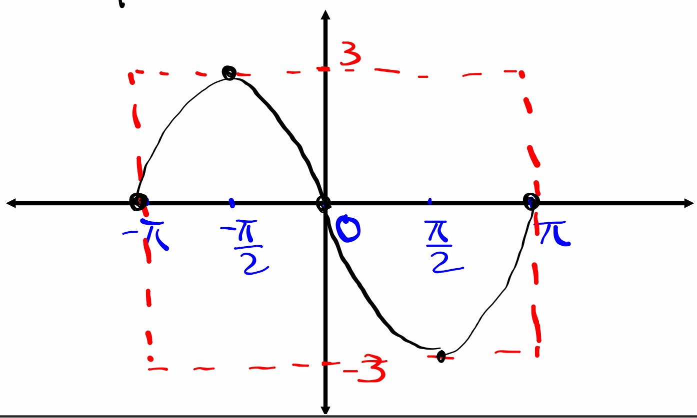
$-\pi \le x \le \pi$

$\frac{-\pi+\pi}{2} = \frac{0}{2} = 0$

$\frac{-\pi+0}{2} = \frac{-\pi}{2}$

$\frac{\pi+0}{2} = \frac{\pi}{2}$

x	$-\pi$	$-\pi/2$	0	$\frac{\pi}{2}$	π
$x+\pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x+\pi)$	0	1	0	-1	0
$3\sin(x+\pi)$	0	3	0	-3	0



Graph $y = \frac{1}{2}\sin(2x - \frac{\pi}{4})$

$0^\circ \le x \le 360^\circ$

$0^\circ \le 2x - 45^\circ \le 360^\circ$

$45^\circ \le 2x \le 405^\circ$

$\frac{45^\circ}{2} \le x \le \frac{405^\circ}{2}$

$22.5^\circ \le x \le 202.5^\circ$

	$\frac{\pi}{8}$		$\frac{5\pi}{8}$		$9\pi/8$
x	22.5°	67.5°	112.5°	157.5°	202.5°
$2x-45^\circ$	0	90°	180°	270°	360°
$\sin(2x-45^\circ)$	0	1	0	-1	0
$\frac{1}{2}\sin(2x-45^\circ)$	0	$\frac{1}{2}$	0	$\frac{-1}{2}$	0

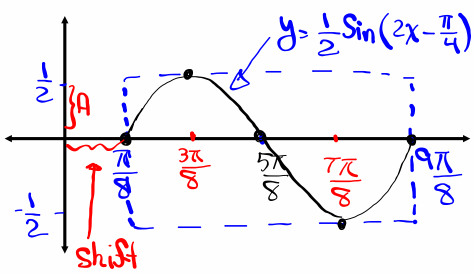
$\frac{22.5^\circ + 202.5^\circ}{2} = 112.5^\circ$

$\frac{22.5^\circ + 112.5^\circ}{2} = 67.5^\circ$

$\frac{202.5^\circ + 112.5^\circ}{2} = 157.5^\circ$

$22.5^\circ = \frac{45^\circ}{2} = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}$

$202.5^\circ = \frac{405^\circ}{2} = \frac{360^\circ + 45^\circ}{2} = \frac{2\pi + \frac{\pi}{4}}{2} = \frac{\frac{9\pi}{4}}{2} = \frac{9\pi}{8}$



Graph $y = 1 + \sin\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

$0 \le x \le 2\pi$

$0 \le \frac{1}{2}x + \frac{\pi}{2} \le 2\pi$

$0 \le x + \pi \le 4\pi$

$-\pi \le x \le 3\pi$

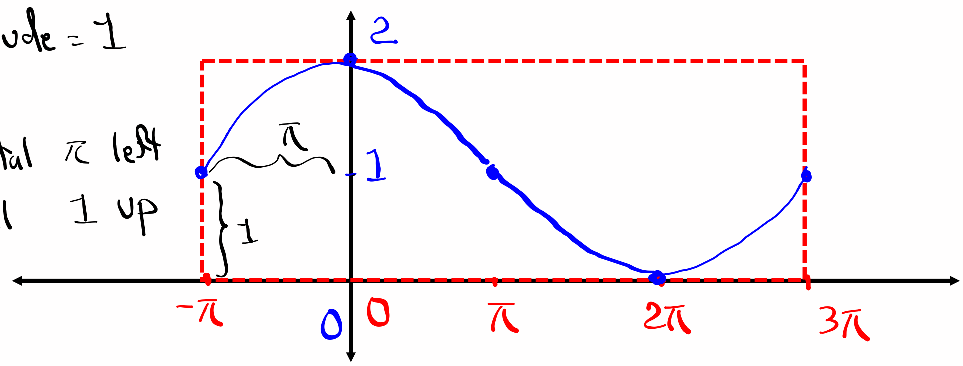
x	$-\pi$	0	π	2π	3π
$\frac{1}{2}x+\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\left(\frac{1}{2}x+\frac{\pi}{2}\right)$	0	1	0	-1	0
$1+\sin\left(\frac{1}{2}x+\frac{\pi}{2}\right)$	1	2	1	0	1

Amplitude = 1

Shift:

Horizontal π left

Vertical 1 up



New formulas:

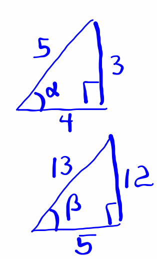
$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Suppose $\sin\alpha = \frac{3}{5}$, α is in QI

$\cos\beta = -\frac{5}{13}$, β is in QII



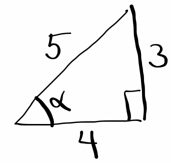
Find $\sin(\alpha+\beta)$

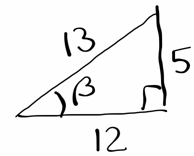
$= \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$= \frac{3}{5} \cdot \frac{-5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{-15}{65} + \frac{48}{65} = \boxed{\frac{33}{65}}$ ✓

$\tan\alpha = \frac{-3}{4}$, α in QII

$\cos\beta = \frac{12}{13}$, β in QIV





Find $\cos(\alpha+\beta)$

$$= \overset{-}{\cos\alpha}\,\overset{+}{\cos\beta} - \overset{+}{\sin\alpha}\,\overset{-}{\sin\beta}$$

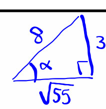
$$= \frac{-4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{-5}{13} = \frac{-48}{65} + \frac{15}{65} = \boxed{\frac{-33}{65}}$$

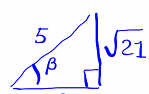
$\sin\alpha = \frac{3}{8}$, $\frac{\pi}{2} < \alpha < \pi$ QII

90° 180°

$\cos\beta = -\frac{2}{5}$ $\pi < \beta < \frac{3\pi}{2}$ QIII

180° 270°





Find $\tan(\alpha+\beta)$

$$= \frac{\overset{-}{\tan\alpha} + \overset{+}{\tan\beta}}{1 - \overset{-}{\tan\alpha}\,\overset{+}{\tan\beta}} = \frac{\frac{-3}{\sqrt{55}} + \frac{\sqrt{21}}{2}}{1 - \frac{-3}{\sqrt{55}}\cdot\frac{\sqrt{21}}{2}}$$

LCD: $2\sqrt{55}$

$$= \frac{-6 + \sqrt{55}\sqrt{21}}{2\sqrt{55} + 3\sqrt{21}}$$

$$= \frac{(-6+\sqrt{55}\sqrt{21})(2\sqrt{55}-3\sqrt{21})}{\underbrace{(2\sqrt{55}+3\sqrt{21})}_{A+B}\underbrace{(2\sqrt{55}-3\sqrt{21})}_{A-B}} = \frac{(-6+\sqrt{55}\sqrt{21})(2\sqrt{55}-3\sqrt{21})}{\underbrace{(2\sqrt{55})^2 - (3\sqrt{21})^2}_{A^2-B^2}}$$

$$= \frac{(-6+\sqrt{55}\sqrt{21})(2\sqrt{55}-3\sqrt{21})}{4\cdot 55 - 9\cdot 21} = \frac{(-6+\sqrt{55}\sqrt{21})(2\sqrt{55}-3\sqrt{21})}{31}$$

$$= \frac{-12\sqrt{55} + 18\sqrt{21} + 2\cdot 55\sqrt{21} - 3\cdot 21\sqrt{55}}{31}$$

$$= \boxed{\frac{-75\sqrt{55} + 128\sqrt{21}}{31}}$$

Prove $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide everything by $\cos A \cos B$

$$= \frac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\dfrac{\cancel{\cos A \cos B}}{\cancel{\cos A \cos B}} - \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\overbrace{\dfrac{\sin A}{\cos A}}^{\tan A} + \overbrace{\dfrac{\sin B}{\cos B}}^{\tan B}}{1 - \underbrace{\dfrac{\sin A}{\cos A}}_{\tan A} \cdot \underbrace{\dfrac{\sin B}{\cos B}}_{\tan B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Find the exact value of $\cos 75°$.

$75° = 45° + 30°$

$$\cos 75° = \cos(\underset{A}{45°} + \underset{B}{30°})$$

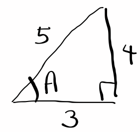
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

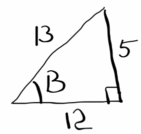
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$



$\sin A = \frac{4}{5}$ A is in QI



$\sin B = \frac{5}{13}$ B is in QII

Find $\cos(A-B) = \overset{+}{\cos A}\ \overset{-}{\cos B} + \overset{+}{\sin A}\ \overset{+}{\sin B}$

$$= \frac{3}{5} \cdot \frac{-12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{-36}{65} + \frac{20}{65} = \boxed{\frac{-16}{65}}$$

Find exact value for $\sin 15°$.

Hint: $15° = 45° - 30°$

$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

$\cos 75° = \sin 15°$

Angles of cofunctions are complementary

Prove $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Recall $\tan(-\alpha) = -\tan\alpha$

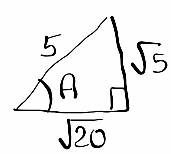
$$\tan(A-B) = \tan(A + (-B))$$

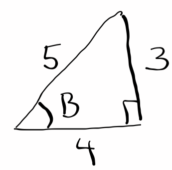
$$= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$= \frac{\tan A + -\tan B}{1 - \tan A \cdot -\tan B} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Given: $\sin A = \dfrac{\sqrt{5}}{5}$, A is in QI,

$\tan B = \dfrac{3}{4}$, B is in QI





Find $\cot(A-B)$

First $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{\sqrt{5}}{\sqrt{20}} - \frac{3}{4}}{1 + \frac{\sqrt{5}}{\sqrt{20}} \cdot \frac{3}{4}} = \frac{\frac{1}{2} - \frac{3}{4}}{1 + \frac{1}{2} \cdot \frac{3}{4}}$$

$$\frac{\sqrt{5}}{\sqrt{20}} = \frac{\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}\sqrt{4}} = \frac{1}{2}$$

LCD = 8

$$= \frac{\overset{4}{\cancel{8}} \cdot \frac{1}{\cancel{2}} - \overset{2}{\cancel{8}} \cdot \frac{3}{\cancel{4}}}{8 \cdot 1 + \cancel{8} \cdot \frac{1}{\cancel{2}} \cdot \frac{3}{\cancel{4}}} = \frac{4 - 6}{8 + 3} = \frac{-2}{11}$$

$$\boxed{\cot(A-B) = \frac{-11}{2}}$$

Verify

$$\boxed{\sin(x+90^\circ)} - \boxed{\sin(x-90^\circ)} = 2\cos x$$

$$= \underbrace{\sin x \cos 90^\circ}_{0} + \underbrace{\cos x \sin 90^\circ}_{\cos x} \qquad \sin x \cos 90^\circ \to 0, \quad \sin 90^\circ \to 1$$

$$= \cos x$$

$$= \underbrace{\sin x \cos 90^\circ}_{0} - \underbrace{\cos x \sin 90^\circ}_{-\cos x}$$

$$= -\cos x$$

$$\sin(x+90^\circ) - \sin(x-90^\circ) = \cos x - (-\cos x) =$$

$$= \cos x + \cos x$$

$$= \boxed{2\cos x}$$

Verify

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B \checkmark$$

$$\cos(A+B) + \cos(A-B) =$$

$$\cos A\cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B} = 2\cos A \cos B$$

Simplify $\cos(90^\circ - x) = \underbrace{\cancel{\cos 90^\circ}\cos x}_{0} + \underbrace{\cancel{\sin 90^\circ}\sin x}_{\sin x}$

$$\cos(90^\circ - x) = \sin x$$

Cofunctions

Complementary

$$90^\circ - x + x = 90^\circ \checkmark$$

Verify

$$\sec(A+B) = \frac{\cos(A-B)}{\cos^2 A - \sin^2 B}$$

$$\sec(A+B) = \frac{1}{\cos(A+B)} = \frac{1}{\cos A\cos B - \sin A\sin B}$$

Multiply everything by $\frac{\cos A}{\cos B}$

$$= \frac{\frac{\cos A}{\cos B}\cdot 1}{\frac{\cos A}{\cancel{\cos B}}\cdot \cos A\cancel{\cos B} - \frac{\cos A}{\cos B}\cdot \sin A\sin B}$$

Multiply by $\cos B$

$$= \frac{\frac{\cos A}{\cos B}}{\cos^2 A - \frac{\cos A\sin A\sin B}{\cos B}} = \frac{\cos A}{\cos^2 A\cos B - \cos A\sin A\sin B}$$

$$= \frac{\cancel{\cos A}}{\cancel{\cos A}(\cos A\cos B - \sin A\sin B)} = \frac{1}{\cos(A+B)}$$

Work on RHS

$$\frac{\cos(A-B)}{\cos^2 A - \sin^2 B} = \frac{\cos A\cos B + \sin A\sin B}{\cos^2 A - \sin^2 B}$$

$$= \frac{\cos A\cos B + \sin A\sin B}{(\cos A + \sin B)(\cos A - \sin B)}$$

Divide by $\cos A\cos B + \sin A\sin B$

$$= \frac{1}{\frac{(\cos A + \sin B)(\cos A - \sin B)}{\cos A\cos B + \sin A\sin B}}$$

$$= \frac{1}{\frac{\cos^2 A - \sin^2 B}{\cos A\cos B + \sin A\sin B}}$$

How about

$$\frac{1}{\cos(A+B)} = \frac{\cos(A-B)}{\cos^2 A - \sin^2 B}$$

Cross-Multiply

$$\cos^2 A - \sin^2 B = \cos(A+B)\cdot\cos(A-B)$$

$$\cos(A+B)\cdot\cos(A-B) =$$

$$[\cos A\cos B - \sin A\sin B][\cos A\cos B + \sin A\sin B] =$$

$$(\cos A\cos B)^2 - (\sin A\sin B)^2 =$$

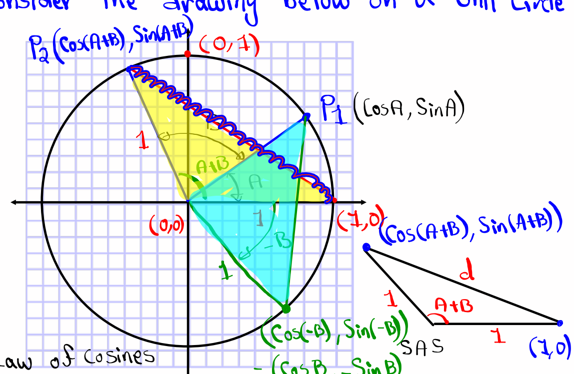
$$\cos^2 A\cdot\cos^2 B - \sin^2 A\sin^2 B =$$

$$\cos^2 A\cdot(1-\sin^2 B) - (1-\cos^2 A)\cdot\sin^2 B =$$

$$\cos^2 A - \cancel{\cos^2 A\sin^2 B} - \sin^2 B + \cancel{\cos^2 A\sin^2 B} =$$

$$\cos^2 A - \sin^2 B$$

Consider the drawing below on a Unit Circle



Law of Cosines

$$d^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cos(A+B)$$

$$d^2 = 2 - 2\cos(A+B)$$

Using distance formula $d = \sqrt{(\cos(A+B)-1)^2 + (\sin(A+B)-0)^2}$

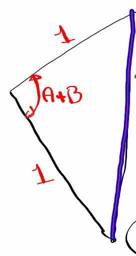
square both sides

$$d^2 = (\cos(A+B)-1)^2 + \sin^2(A+B)$$

$$= \cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B)$$

$$= 1 - 2\cos(A+B) + 1$$

$$= 2 - 2\cos(A+B)$$



$(\cos A, \sin A)$

$$d = \sqrt{(\cos A - \cos B)^2 + (\sin A - {-\sin B})^2}$$

$$= \sqrt{(\cos A - \cos B)^2 + (\sin A + \sin B)^2}$$

$$= \sqrt{\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B}$$

$(\cos(-B), \sin(-B)) = (\cos B, -\sin B)$

$$= \sqrt{1 + 1 - 2\cos A\cos B + 2\sin A\sin B}$$

$$d = \sqrt{2 - 2\cos A\cos B + 2\sin A\sin B}$$

$$\underbrace{\cancel{2} - 2\cos(A+B)}_{d^2} = \cancel{2} - 2\cos A\cos B + 2\sin A\sin B$$

Divide by -2

$$\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$$

Replace B with $-B$,

$$\cos(A - B) = \cos A\cos(-B) - \sin A\sin(-B)$$

$$= \cos A \cdot \cos B \ \cancel{-} \sin A \cdot -\sin B$$

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

Prove $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\sin(A+B) = \cos(90^\circ - (A+B))$$

Cofunctions

$A+B$ + ___ $= 90^\circ$

___ $= 90 - (A+B)$

$$= \cos 90^\circ \cos(A+B) + \sin 90^\circ \sin(A+B)$$

0 1

$$= \sin(A+B)$$

$$\sin(A+B) = \cos(90^\circ - (A+B))$$

$$= \cos(\boxed{90^\circ - A} - \boxed{B})$$

$$= \underbrace{\cos(90^\circ - A)}\cdot\cos B + \underbrace{\sin(90^\circ - A)}\cdot\sin B$$

$$= \boxed{\sin A}\cos B + \boxed{\cos A}\sin B$$

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$

Replace B with $-B$

$$\sin(A - B) = \sin A\,\boxed{\cos(-B)} + \cos A\,\boxed{\sin(-B)}$$

$\cos B$ $-\sin B$

$$= \sin A\cos B - \cos A\sin B$$